Sheet 1

Handed out on 26. 10. 17 for the Tutorial on 09. 11. 17

Problem 1: Planck's law and Rayleigh-Jeans law (4P)

In statistical physics a system with the eigenenergies ϵ_k is populated in thermodynamic equilibrium according to a Boltzmann distribution, which results in a fractional population of each possible state k of

$$f_k = \frac{d_k}{Z} e^{-\frac{\epsilon_k}{k_B T}} \ . \tag{1}$$

Therein, d_k is the degeneracy of the level k and

$$Z = \sum_k d_k e^{-\frac{\epsilon_k}{k_B T}} \tag{2}$$

is the partition function of the system that normalizes the total population probability to $\sum_k f_k = 1$.

For an electromagnetic field of frequency ν, Planck proposed that each mode can be occupied by a whole number of photons $N \in \mathbb{N}$, and he therefore concluded that the energy of this mode is given by

$$E_N = N h \nu \ . \tag{3}$$

(a) Calculate the average energy $\langle E \rangle$ of an electromagnetic mode in thermal equilibrium at a mode temperature T. (2P)
Hint: The following mathematical trick may be helpful:

$$\sum_\alpha \alpha e^{\alpha\beta} = \sum_\alpha \frac{\partial}{\partial \beta} e^{\alpha\beta} = \frac{\partial}{\partial \beta} \sum_\alpha e^{\alpha\beta} \ . \tag{4}$$

(b) Using the electromagnetic properties of space presented in the lecture, show that Planck's formula can also be derived from the result of (a). (1P)

This actually was the original deduction of that formula, performed by Planck some years before Einsteins derivation from the postulated stimulated emission.
(c) In statistical thermodynamics it is well known that in thermal equilibrium each eigenoscillation of a harmonic system will have an average energy of

$$\epsilon = k_B T \ . \tag{5}$$

Use this result to derive the Rayleigh-Jeans law. (1P)

Problem 2: Stefan-Boltzmann's law (4P)

Using Planck's law it is possible to derive the total energy of a hohlraum of volume V when the walls are held at a constant temperature T.

(a) Calculate the total energy density of the radiation in the hohlraum. (1P)
(b) Show that a small hole in one of the hohlraum's walls will radiate a total thermal intensity (2P)

$$I(T) = \sigma T^4 \; . \tag{6}$$

(c) Derive the value of the proportionality constant σ in SI units. (1P)
Hint: You will need the identity

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15} \; . \tag{7}$$

Problem 3: Wien's displacement law (4P)

Using Planck's law it is possible to derive Wien's displacement law, stating that the wavelength of the maximum spectral energy density times the corresponding temperature

$$\lambda_{max} T = \kappa = \text{const.} \tag{8}$$

is constant.
(a) Perform this derivation. (1P)
(b) Calculate the constant κ in SI units. (1P)
Hint: You will need the solution of

$$e^{-x} = 1 - \frac{x}{5} \quad \Rightarrow \quad x = 0 \quad \vee \quad x \approx 4.96511 \; . \tag{9}$$

(c) The solar radiation propagates $\approx 1.5 \times 10^{11}$ m to reach the earth, where its incident intensity is measured to $1.36 \; \frac{\text{kW}}{\text{m}^2}$ (solar constant). The sun's radius is $\approx 7 \times 10^8$ m. Calculate the sun's peak thermal emission wavelength. (2P)
Hint: You need the result of problem 2(b+c).